



8.10 Linear operators

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Introduction

- In this topic, we will
 - Review the properties of linear maps with composition
 - Discuss linear operators with composition
 - Discuss the identity operator
 - Define matrix powering
 - Discuss consequences of noncommutativity
 - Describe matrix polynomials
 - Define inverses





Composition of linear maps

- The collection of all linear operators $\mathcal{L}(\mathcal{U})$ has scalar multiplication, operator addition, and operator composition, and thus has the following properties:
 - The collection forms a vector space under scalar multiplication and operator addition
 - The additional axioms associated with operator composition include:
 11. Composition is associative: $C(BA) = (CB)A$
 12. Composition distributes over operator addition from the left
$$B(A_1 + A_2) = BA_1 + BA_2$$
 13. Composition distributes over operator addition from the right
$$(B_1 + B_2)A = B_1A + B_2A$$
 14. Composition is bilinear: $\alpha(BA) = (\alpha B)A = B(\alpha A)$





Composition of linear maps

- Operators, however, have one additional axiom:
 15. There is an identity operator I such that $AI = A = IA$ for all linear operators A

Theorem

The linear operator I defined as $I\mathbf{u} = \mathbf{u}$ serves as an identity operator.

Proof:

$$\begin{aligned}(AI) \mathbf{u} &= A(I \mathbf{u}) \\ &= A\mathbf{u} \\ &= I(A\mathbf{u}) \\ &= (IA) \mathbf{u} \quad \blacksquare\end{aligned}$$

– Note that this means I commutes with all linear operators A





Algebra of linear operators

- A vector space together with a composition that satisfies the five additional axioms we saw define a *unital associative algebra*
 - Thus, the focus of this course will be looking at the
Unital associative algebra of
linear operators mapping a vector space to itself
 - Hence the name of the course: linear algebra
- Generally, however, we will continue to refer to $\mathcal{L}(\mathcal{U})$ as the vector space of linear operators mapping a vector space to itself
 - It is assumed it has a composition of operators
and an identity operator





Composition of linear maps

- For $\mathcal{L}(\mathbf{R}^n)$, these operators can be represented as $n \times n$ matrices
 - The identity operator is the $n \times n$ diagonal matrix:

$$I_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$





Invertibility

- Recall that the multiplicative inverse of 1 is 1
 - Notice that $II = I$?
 - Hence, we have found the compositional inverse of I is itself
 - We will simply call this the multiplicative inverse
- For any other $A \in \mathcal{L}(\mathcal{U})$,
if there exists an $B \in \mathcal{L}(\mathcal{U})$ such that $AB = BA = I$,
then we will say A is invertible and
denote its multiplicative inverse as A^{-1}





Composition of linear maps

- There are two additional field axioms that are not satisfied by linear operators:
 - Operator composition is not commutative
 - Not all linear operators have compositional inverses





Composition of linear maps

- For example, we note

$$\begin{pmatrix} 0 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 2 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix}$$

- Also note that

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

- However, $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 1.5 & -0.5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\begin{pmatrix} -2 & 1 \\ 1.5 & -0.5 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$





Powers of operators

- We can also define powers of operators:

For $A \in \mathcal{L}(\mathcal{U})$ and a non-negative integer n , define

$$A^n = \begin{cases} I & n = 0 \\ AA^{n-1} & n > 0 \end{cases}$$

For an invertible $A \in \mathcal{L}(\mathcal{U})$ and an integer n , define

$$A^n = \begin{cases} (A^{-1})^{-n} & n < 0 \\ I & n = 0 \\ AA^{n-1} & n > 0 \end{cases}$$





Powers of operators

Theorem

If $A \in \mathcal{L}(\mathcal{U})$ and m and n are non-negative integers,
then $A^{m+n} = A^m A^n$

Theorem

If $A \in \mathcal{L}(\mathcal{U})$, A is invertible and m and n are integers,
then $A^{m+n} = A^m A^n$

Theorem

If $A \in \mathcal{L}(\mathcal{U})$ and m and n are non-negative integers,
then $(A^m)^n = A^{mn}$

Theorem

If $A \in \mathcal{L}(\mathcal{U})$, A is invertible and m and n are integers,
then $(A^m)^n = A^{mn}$





Powers of operators

- On the other hand, none of these common simplifications work:

$$(AB)^2 \neq A^2 B^2$$

$$(AB)^2 = ABAB$$

$$(A + B)^2 \neq A^2 + 2AB + B^2$$

$$(A + B)^2 = A^2 + AB + BA + B^2$$

- However, we still have

$$(\alpha A)^n = \alpha^n A^n$$

- This is more familiar, but is a consequence of $AI = A$:

$$(A + I)^n = A^n + \binom{n}{n-1} A^{n-1} + \binom{n}{n-2} A^{n-2} + \cdots + \binom{n}{2} A^2 + \binom{n}{1} A + I$$

$$(x + 1)^n = x^n + \binom{n}{n-1} x^{n-1} + \binom{n}{n-2} x^{n-2} + \cdots + \binom{n}{2} x^2 + \binom{n}{1} x + 1$$





Efficient calculations of integer powers

- If A is a matrix, the inefficient way to calculate A^n is to start with A and multiply that by A a total of $n - 1$ times
- This is much more efficient:

$$A^n = \begin{cases} I & n = 0 \\ A & n = 1 \\ (A^k)^2 & n = 2k \\ A(A^k)^2 & n = 2k + 1 \end{cases}$$

- If n is even, it requires one matrix-matrix multiplication
- If n is odd, it requires two





Efficient calculations of integer powers

- For example, to calculate A^{49} , we must:
 - Calculate A^{24} , square and multiply by A
- To calculate A^{24} , we must:
 - Calculate A^{12} and square it
- To calculate A^{12} , we must:
 - Calculate A^6 and square it
- To calculate A^6 , we must:
 - Calculate A^3 and square it
- To calculate A^3 , we must:
 - Retrieve A , square it and multiply by A
- This is a total of 7 matrix-matrix multiplications, and not 48 such multiplications

$$A^n = \begin{cases} I & n = 0 \\ A & n = 1 \\ (A^k)^2 & n = 2k \\ A(A^k)^2 & n = 2k + 1 \end{cases}$$





Polynomials on operators

- Now that we have defined powers of linear operators,
we can now define polynomials on linear operators
- Let A be a linear operator,
then if we define p to be the polynomial such that

$$p(A) = A^2 + 2A + 6I,$$

we now have a polynomial in A

- If the linear operators are real, the coefficients must be from $\mathbf{R}$
- If the linear operators are complex,
the coefficients can come from $\mathbf{C}$





Polynomials on operators

- For example, given p defined as $p(A) = A^2 + 2A - 6I$,

and $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, we note that

$$A^2 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix}$$

$$2A = 2 \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix}$$

$$-6I = -6 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -6 & 0 \\ 0 & -6 \end{pmatrix}$$

- Thus, $p(A) = A^2 + 2A - 6I = \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix} + \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix} + \begin{pmatrix} -6 & 0 \\ 0 & -6 \end{pmatrix}$
 $= \begin{pmatrix} 3 & 14 \\ 21 & 24 \end{pmatrix}$





Summary

- Following this topic, you now
 - Know about the application of the identity operator
 - Understand about the existence of inverses
 - Know that composition is not commutative and not all operators have compositional inverses
 - Know how to find powers of matrices
 - Understand that you can define polynomials on linear operators
- Important:
 - From here on in, we will refer to the composition of linear operators as multiplication, and the compositional inverse as the multiplicative inverse





References

- [1] https://en.wikipedia.org/wiki/Linear_map#Forming_new_linear_maps_from_given_ones
- [2] https://en.wikipedia.org/wiki/Matrix_multiplication





Acknowledgments

None so far.





Colophon

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The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

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